APPROXIMATE CALCULATION OF THE SOLIDIFICATION OF A BINARY MIXTURE IN AXISYMMETRIC ENCLOSURES

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We shall study the dynamics of the redistribution of the stress–strain state and the temperature field under the conditions of solidification of a binary mixture in axisymmetric enclosures. The enclosures are represented by a cylindrical cavity in an infinite block, a cylindrical channel bounded by an elastic shell, and a ring-shaped channel, formed between two elastic shells with different radius. Cooling temperature conditions are created in the medium outside the enclosure. The solidification front moves from the outer radial contour of the enclosure toward the symmetry axis, displacing the impurity in the liquid phase with some coefficient of distributivity. The density of the liquid phase drops in a jump-like manner at the solidification front, which raises the pressure in it and causes a redistribution of the stresses in the medium. We assume that the temperature of the phase transition on the solidification front is a function of the instantaneous values of the pressure and concentration of the impurity. Problems of this type arise in modern technology in the calculation of the dynamics of solidification of materials, in which the shear stresses relax comparatively quickly and have virtually no effect on the magnitude of the pressure developing in the liquid phase and reaching values which are equilibrium on the phase diagram of the solidification process. An example of such solidification is the freezing of brine, when the freezing front forces the dissolved salt impurity back into the liquid phase, increasing the concentration in it, with pressure levels of several hundreds of megapascals being developed. We shall study the elastoplastic model of the mechanical behavior of a solid mass and the elastic model for shells. We shall solve the problem in a planar axisymmetric formulation, in which the stress, temperature, concentration, and displacement fields are independent of the coordinate parallel to the axis of the enclosure.

In accordance with the geometry and physical content of the problem, we distinguish three different zones according to their heat and mass transfer and mechanical characteristics with the parameters indexed correspondingly as $i = 1, 2$, and 3. The first zone contains the liquid phase of the mixture ($a \leq r \leq s(\tau)$), the second ($s(\tau) \leq r \leq b$) is the solid phase of the mixture, and the third ($b \leq r$) occurs for solidification in a cavity of some solid mass, behaving in an elastoplastic manner under a load. In the case of enclosures in the form of shells, there is no third zone; the contour $r = b$ is the radius of the external shell and the contour $r = a$ can be the radius of the internal shell; when solidification occurs within a single cylindrical shell we obtain the solution by setting $a = 0$. We shall write down the corresponding system of equations for heat transfer, diffusion of the impurity, and the stress–strain state assuming that a constant cooling temperature T_m is given throughout the entire process either on the contour of the external shell or at a point infinitely far away from the cavity in the solid mass.

The system of heat transfer equations has the form

$$r\frac{\partial T_i}{\partial\tau} = a_i\frac{\partial}{\partial r}\left(r\frac{\partial T_i}{\partial r}\right),\ i = 1, 2,\ \lambda_2\frac{\partial T_2}{\partial r}\Big|_{r=s} - \lambda_1\frac{\partial T_1}{\partial r}\Big|_{r=s} = l\rho_2\frac{ds}{d\tau},$$

$$T_2(s, \tau) = T_1(s, \tau) = T_{\text{p}}(\tau),\ T_{\text{p}}(\tau_0) = T_{\text{p},0},\ T_2(b, \tau) = T_3(b, \tau),$$

$$\lim_{r\to\infty} T_3(r, \tau) = T_{\text{м}} \quad \text{or} \quad T_2(b, \tau) = T_{\text{м}},$$

$$s(\tau_0) = b,\ T_i(\tau_0) = T_{i0}(r).$$

Here T_i is the temperature; a_i and λ_i are the coefficients of thermal diffusivity and thermal conductivity; ℓ is the latent heat; ρ_i is the density; τ and τ_0 are the instantaneous time and the initial time; T_{i0} is the initial temperature distribution; s is the coordinate

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of the solidification front; and T_p and T_{po} are the instantaneous and initial values of the phase-transition temperature.

We assume that the redistribution of the impurity occurs only in the liquid and solid phases of the mixture in the enclosure, so that the corresponding equations of diffusion are written down only in the first and second zones:

$$r\frac{\partial c_i}{\partial \tau} = D_i \frac{\partial}{\partial r}\left(r\frac{\partial c_i}{\partial r}\right), \quad i = 1,2,$$
$$D_2\frac{\partial c_2}{\partial r}\bigg|_{r=s} - D_1\frac{\partial c_1}{\partial r}\bigg|_{r=s} = \dot{s}\,(c_2 - c_1)_{r=s},$$
$$c_2(s, \tau) = kc_1(s, \tau), \; c_1(r, \tau_0) = c_0,$$
$$\frac{\partial c_1}{\partial r}\bigg|_{r=a} = \frac{\partial c_2}{\partial r}\bigg|_{r=b} = 0,$$

where c_i and D_i are the concentration and diffusion coefficients; c_0 is the initial concentration of the impurity in the liquid phase, calculated in terms of the mass concentration x; and k is the distribution factor of the impurity [1].

The equations for the stress–strain state of the problem of solidification, neglecting diffusion of the impurity, are studied in [2], where T_p is a function of the pressure in the liquid phase p, unlike this work where T_p is also a function of the impurity concentration c_i. We shall use the solution presented in [2] for the indicated equations, obtained assuming a hydrostatic distribution of stresses in the solidifying mixture and having the form of the equation relating the pressure p and the position of the solidification front s.

In the case of solidification in the space between two thin-walled cylindrical shells we write the indicated mixture in the explicit form

$$p = p_{\text{in}} + \left[\alpha_V\left(1 - \frac{s^2}{b^2}\right) + \frac{2\lambda_{\text{ext}}\sigma_{\text{ext}}}{b}\left(1 + \frac{a\lambda_a\sigma_a}{b\lambda_{\text{ext}}\sigma_{\text{ext}}}\right)\right] \Big/ \left\{\frac{2\lambda_{\text{ext}}}{b}\left(1 + \frac{a\lambda_a}{b\lambda_{\text{ext}}}\right) + \right.$$
$$\left. + \frac{1}{k_2}\left[\frac{k_2}{k_1}\left(1 - \frac{a^2}{s^2}\right)\frac{s^2}{b^2} + 1 - \frac{s^2}{b^2}\right]\right\}, \tag{1}$$

where $\lambda_a = 3a^2/(4E_a\delta_a)$; $\lambda_{\text{ext}} = 3b^2/(4E_{\text{ext}}\delta_{\text{ext}})$; E_a, E_{ext} and δ_a, δ_{ext} are the moduli of elasticity and the thicknesses of the inner and outer shells; k_1 and k_2 are the bulk moduli of elasticity of the liquid and solid phases of the mixtures; $\alpha_V = 1 - \rho_2/\rho_1$ is the coefficient of volume deformation accompanying solidification ($\alpha_V > 0$); σ_a, σ_{ext} are the known internal pressure on the inner shell and the external pressure on the external shell; and p_{in} is the known internal initial pressure in the space between the shells.

When solidification occurs in the cylindrical cavity in an elastoplastic solid mass we write out the above-mentioned relationship following [2]. We assume that the mixture solidifies in the volume formed by the radial contour of the cavity and the surface of the interior elastic shell with smaller radius r = *a*.

At the elastic stage of deformation of the solid mass the explicit relationship between p and s has the form

$$p = p_{\text{in}} + \left[2(1 - \nu_3)\frac{\sigma_\infty}{G_3} + \alpha_V\left(1 - \frac{s^2}{b^2}\right) + 2a\lambda_a\frac{\sigma_a}{b^2}\right] \Big/ \left\{\frac{1}{G_3} + \frac{2a\lambda_a}{b^2} + \right.$$
$$\left. + \frac{1}{k_2}\left[\frac{k_2}{k_1}\left(1 - \frac{a^2}{s^2}\right)\frac{s^2}{b^2} + 1 - \frac{s^2}{b^2}\right]\right\}. \tag{2}$$

Here σ_∞ is the radial stress at infinity; $G_3 = E_3/[2(1 - \nu_3)]$; E_3, ν_3 are the modulus of elasticity and the Poisson ratio of the solid mass.

At the stage of elastoplastic deformation of the solid mass, occurring after the pressure reaches a value of p_g, the relationship between the pressure and the solidification front can be written as a transcendental equation

$$p = 2(1-\nu_3)\sigma_\infty + \tau_3\left(1 + \ln\left(\frac{G_3}{\tau_3}\left(\alpha_V\left(1-\frac{s^2}{b^2}\right) + \frac{2a\lambda_a\sigma_a}{b^2} + \right.\right.\right. \tag{3}$$
$$\left.\left.\left. + p\left\{\frac{2a\lambda_a}{b^2} + \frac{1}{k_2}\left[\frac{k_2}{k_1}\left(1-\frac{a^2}{s^2}\right)\frac{s^2}{b^2} + 1 - \frac{s^2}{b^2}\right]\right\}\right)\right)\right) + p_{\text{in}},$$

where $p_g = 2(1 - \nu_3) + \tau_3$; τ_3 is the creep parameter of the mass for Mies' condition of plasticity.

To close the system of equations presented for the unknowns $T_i(\tau)$, $c_i(\tau)$, $T_p(\tau)$, $s(\tau)$ and $p(\tau)$ it is necessary to add a relation between the phase-transition temperature and the pressure and concentration. This equation can be obtained from the condition that the chemical potentials are equal at the interface:

$$M\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)dp - Ml\frac{dT_p}{T_p} + RdT_p\ln\frac{1-c_1}{1-c_2} - RT_p\left(\frac{dc_1}{1-c_1} - \frac{dc_2}{1-c_2}\right) = 0,$$

where M is the mass of a kilomole of solvent and R is gas constant.

The complete system of equations presented above can be solved numerically using an appropriate algorithm, but this is a quite laborious path to the quantitative results. The mutual effect of the temperature, impurity concentration, and stress—strain fields taking into account the pressure and concentration dependence of T_p can be determined by simplifying the starting system. The heat transfer part of the system is substantially simplified by assuming that there is no temperature gradient in the liquid phase and the temperature distribution in the solidified zone is quasistationary. The diffusion part of the system is simplified by assuming that there are no concentration gradients in either phase; we shall assume that all of the redistribution occurs on the solidification front according to the equation of balance of concentration in the liquid phase $c = c_1(\tau)$

$$c = c_0\left(\frac{b^2-a^2}{s^2-a^2}\right)^{1-k}. \tag{4}$$

If it is assumed also that the mixture is incompressible in both phases $k_1 = k_2 = \infty$ and that the internal shell is rigid for the case of solidification in a cavity $\lambda_a = 0$, then instead of the transcendental equation (3) we shall obtain analogously to (1) and (2) an explicit relationship p(s) at the stage of plastic deformation of the solid mass.

Under the assumptions made the starting system of partial differential equations reduced to a system of two ordinary differential equations in the dimensionless form

$$\frac{dy}{dt} = -\frac{\lambda(\Theta_M - \Theta_p)}{y\ln y}; \tag{5}$$

$$\left(\beta\ln\frac{1-c}{1-kc} - \Theta_p\right)d\Theta_p = \psi_0\alpha_V df + \beta\Theta_p\frac{(1-k)}{(1-c)(1-kc)}dc, \tag{6}$$

where $y = \frac{s}{b}$; $\Theta_p = \frac{T_p}{T_0}$; $\Theta_M = \frac{T_M}{T_0}$; $t = \frac{\tau}{\tau_0}$; $f = \frac{p}{p_0}$; $\lambda = \frac{\lambda_2 T_0\tau_0}{l\rho_2 b^2}$; $\beta = \frac{RT_0}{Ml}$; $\psi_0 = \frac{p_0}{\rho_2 l}$; and τ_0, T_0, and p_0 are the time, temperature, and pressure scales.

We note that under the above-indicated simplifying assumptions we obtain for the parameters c and f entering into the system (5) and (6) the explicit expressions (1)-(4), and the expressions for the pressures (1) and (2) will give quantitatively the same result, if $\sigma_\infty \neq 0$ and the mechanical parameters are related by

$$G_3 = b/(2\lambda_b),\ 2\sigma_\infty(1-\nu_3) = \sigma_b. \tag{7}$$

This relation makes it possible to rescale the examples of the calculation of solidification between shells to the case of solidification in a cavity filled with a mixture in a medium which remains elastic throughout the entire process.

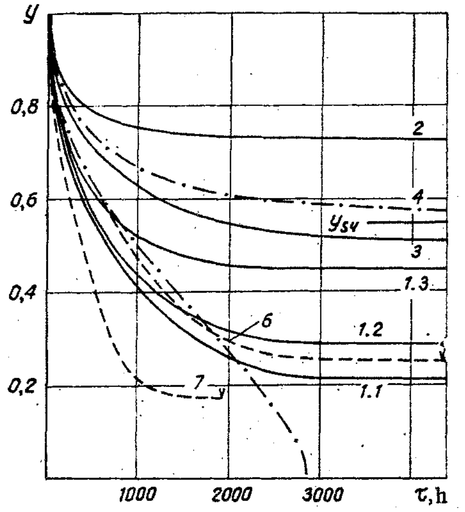

Fig. 1

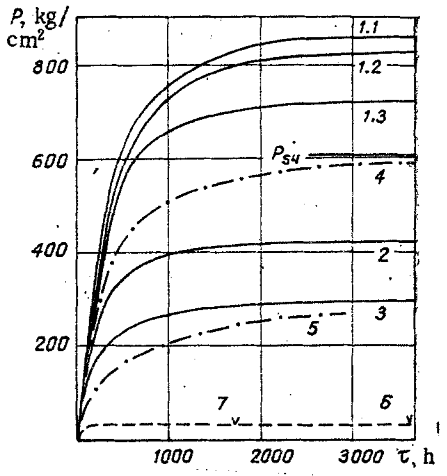

Fig. 2

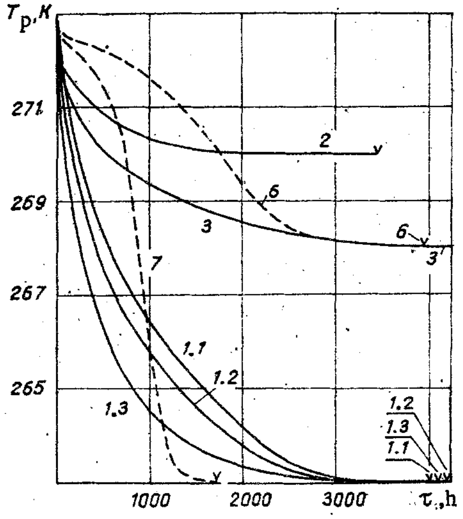

Fig. 3

Approximate calculations using the proposed model are presented for freezing of a water solution of salt NaCl. The following values of the parameters were chosen for the calculations: $\tau_0 = 1$ h, $T_0 = 273$ K, $p_0 = 1$ kg/cm², $k = 0.01$, $R = 8.32 \cdot 10^3$ J/(k·mole), $M = 0.018$ kg/mole, $\ell = 334 \cdot 10^3$ J/kg, $\lambda_2 = 2.21$ W/(m·K), $\rho_1 = 1000$ kg/m³, $\rho_2 = 910$ kg/m³, $\delta_a = \delta_b = 0.01$ m, $E_a = E_b = 2 \cdot 10^6$ kg/cm², $a = 0$, $b = 1$ m, $k_1 = k_2 = \infty$, $\sigma_a = \sigma_b = p_b = \sigma_\infty = 0$.

Figures 1-4 show the results of the calculation of freezing of a salt solution in a cavity for the elastic (solid lines) and elastoplastic (broken lines) models of the solid mass containing the cavity. The dot-dashed curves show the calculation with zero impurity concentration. For comparison, seven variants of the conditions of the problem are presented (curves 1.1-1.3 refer to variant 1, and curves 2-7 refer to the remaining variants). Curves 1.1-1.3 correspond to an initial concentration of x = 2.5, 5, and 15 kg/m³ at $T_M = 263$ K; curves 2 and 3 correspond to $G_3 = 1 \cdot 10^4$ and $0.3 \cdot 10^4$ kg/cm² at $T_m = 268$ K and x = 15 kg/m³, and curves 4 and 5 correspond to the same values of T_M and G_3, but with x = 0. The graphs 6 and 7 were constructed for the elastoplastic model of a solid block containing a cavity for the same values $\tau_3 = 5$ kg/cm², $G_3 = 5 \cdot 10^4$ kg/cm², x = 5 kg/m³ for $T_m = 268$ and 263 K. We note that according to (7) the lines 1-5 correspond also to the case of freezing in a single shell, for which with b = 0.27 m and $E_b = 2 \cdot 10^6$ kg/m² the value $G_3 = 1 \cdot 10^4$ kg/cm² corresponds to $\delta_b = 2 \cdot 10^{-3}$ m, and the value $G_3 = 0.3 \cdot 10^4$ kg/cm² corresponds to $\delta_b = 6.7 \cdot 10^{-4}$ m.

Analysis of the computational results shows that the impurity substantially affects the dynamics of the front s (Fig. 1), the pressure p (Fig. 2), the phase-transition temperature T_p (Fig. 3), and the concentration c (Fig. 4). In the absence of the impurity (x = 0, curves 4 and 5) the process proceeds either to complete solidification (Curve 5 in Fig. 1) or the front approaches a stationary value y_{s4} over an infinite time, while the pressure

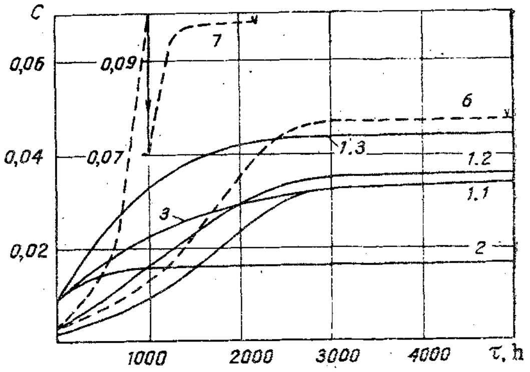

Fig. 4

assumes equilibrium values p_{s4} (curve 4 in Figs. 1 and 2). The calculations for the same values of the cooling temperature T_M and the parameter G_3 but taking into account the impurity give lower pressures (curves 2 and 3 in Fig. 2); in addition, increasing the strength of the solid mass (or shell) raises the pressure level. In the presence of the impurity, first of all, complete solidification is not observed, and all parameters of the process assume stabilized values over a finite time interval at some time denoted by the symbol V on the graphs; second, the pressure for $x \neq 0$ is lower than for $x = 0$ (curves 2-4). Increasing the initial impurity concentration lowers the pressure developed and the volume of the solidified phase, as illustrated by curves 1.1-1.3.

Taking into account the plastic deformations of the solid mass surrounding the cavity substantially reduces the pressures developed, which for two values of T_M have very close graphs as a function of time (curves 6, 7 in Fig. 2), but the stabilized values are achieved at different times. For an elastoplastic model the reduction in T_M accelerates the convergence of the parameters to the stabilized values, increases the volume of the solidified phase (see Fig. 1), and substantially increases the stabilized concentration (curves 6 and 7 in Fig. 4). This strong increase in the stabilized value of the concentration is determined by the acceleration of solidification as T_M decreases, which under real conditions reduces the time over which the impurity concentration fields are equalized in the phases of the mixture. This effect is observed in a quantitatively less distinct form also for an elastic model of the cavity (or the case of a single shell) – this is evident from a comparison of curves 2 and 1.2 in Fig. 4.

In conclusion, we note the following. First of all, the results of the calculations of the solidification in the space between coaxial shells are qualitatively virtually identical to the examples presented here. Second, according to the calculations performed with k varying in the range from $1 \cdot 10^{-2}$ to $1 \cdot 10^{-6}$, its decrease intensifies the forcing of the impurity into the liquid phase, which reduces the magnitude of the stabilized pressure. Third, the calculations performed demonstrated that it is possible to forecast the loading and to estimate the reliability of the enclosures containing the solidifying mass. Fourth, the calculations did not reveal the conditions under which solidification occurs with the existence of a transition phase.

The above-presented model of solidification does not include the possibility of the appearance of a transitional phase, studied in [3, 4]. In order to describe the process of solidification with a transitional phase in the problem studied here the mathematical model must be supplemented. The transitional phase in the problem under study does not appear when the temperature of the solid phase is lower than the phase-transition temperature.

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DEVELOPMENT OF A TEMPERATURE FIELD IN A TURBULENT FLOW WITH UNSTEADY HEAT TRANSFER

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Heat-transfer processes occurring under unsteady conditions are encoutnered fairly often in different areas of modern technology. The theory of unsteady heat transfer in turbulent flows is still far from complete, which has to do with the serious difficulties encountered in attempting to mathematically describe the processes. Calculation of these processes is also considerably complicated by the need to solve a coupled problem, since processes of heat transfer in the heat carrier are inextricably allied with the development of a temperature field in the walls of the channel [1, 2].

The experimental study of unsteady heat transfer in turbulent flows is a rather complicated technical problem, due both to the effect of a large number of parameters on the character of the process and to the need to rapidly collect and analyze a large volume of information. Measurements show that the effect of transience on heat transfer may be substantial [2-4]. At the same time, calculations with models which include the hypothesis of a quasisteady structure for the turbulent flow do not always yield satisfactory results [5]. In connection with this, a deeper understanding of the processes taking place in unsteady heat transfer requires sufficiently complete and reliable information on the velocity and temperature fields in the immediate vicinity of the wall, including the region of the viscous and thermal sublayers.

Here we report results of measurement of the development of the temperature field in a turbulent flow of water with a sudden change in heat release in the channel wall. The measurements were made in the Reynolds number range from 11,200 to 112,000 and embrace the region near the wall, including the viscous sublayer. The tests were conducted on a closed hydrodynamic loop which included a constant-level tank, working section, receiving tank, cooler, and pump. The working section (Fig. 1) was a channel 1 of rectangular cross section measuring 20 × 40 mm and consisting of a hydrodynamic stabilization section 96 diameters long and a heating section 3 which was 36 diameters long. Three of the walls of the heated section were made of organic glass, while the remaining wall (40 mm wide) was made of stainless steel 0.1 mm thick. The steel wall was stuck onto a glass–textolite base. The strip was washed by the flow of working fluid and heated by the passage of an electrical current through it.

The temperature in the flow was recorded with a specially made thermocouple probe 2 of the needle type. The transverse dimension of the hot junction was about 5 μm. The probe was inserted into the flow through the top, unheated wall of the channel 78 cm from the beginning of the heating section. Temperature in the flow was measured relative to the temperature of the cold wall in the test section. This kept the test results from being affected by small fluctuations n the temperature of the working liquid during the tests. The temperature drop between the hot and cold walls of the channel was no greater than 10 K. The absolute error of the measurements of instantaneous temperature was ±0.07 K. More details concerning the design of the probe and the set-up of the experimental unit are available in [6].

The test unit was supplied with heat from a dc generator. The process of unsteady heat transfer caused a sudden change in the amount of electric power supplied to the strip heater, which in turn led to a sharp change in heat release. Results of measurements of static characteristics of temperature pulsations in a turbulent flow during steady heat transfer were published in [7].

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